Convergence of Robust Models

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Mathematics VII: Stochastics

UNIVERSITÄT BAYREUTH

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Outline

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Ideal Model

- parametric family of probability measures

$$\mathcal{P} = \{P_\theta \,|\, \theta \in \Theta\} \qquad \Theta \subset \mathbb{R}^k \text{ (open)}$$

  defined on some measurable space $(\Omega, \mathcal{A})$

- smoothly parameterized; i.e., L_2 differentiable at $\theta \in \Theta$ with L_2 derivative $\Lambda_\theta \in L_2^k(P_\theta)$, $\mathrm{E}_\theta \Lambda_\theta = 0$ and

- Fisher information of full rank

$$\mathcal{I}_\theta = \mathrm{E}_\theta \Lambda_\theta \Lambda_\theta^\tau \qquad \mathcal{I}_\theta \succ 0$$

Influence Curves (ICs)

Definition

The set $\Psi_2(\theta)$ *of all* **square integrable ICs at P_θ** *consists of*

- *all* $\psi_\theta \in L_2^k(P_\theta)$ *which are*
- *centered:* $E_\theta\, \psi_\theta = 0$ *and*
- *Fisher consistent:* $E_\theta\, \psi_\theta \Lambda_\theta^\tau = \mathbb{I}_k$

(where $\mathbb{I}_k$ k-dimensional identity matrix)

AL Estimators

Definition

An asymptotic estimator $S_n\colon (\Omega^n, \mathcal{A}^n) \to (\mathbb{R}^k, \mathbb{B}^k)$ *is called* **asymptotically linear at P_θ** *if there is an*

- *IC* $\psi_\theta \in \Psi_2(\theta)$
- *with*

$$S_n = \theta + \frac{1}{n}\sum_{i=1}^{n} \psi_\theta(y_i) + o_{P_\theta^n}(n^{1/2})$$

Infinitesimal Neighborhoods

Convex contamination neighborhood (gross error model)

$$U_c(\theta, r_n) = \{(1 - r_n)_+ P_\theta + (1 \wedge r_n)\, Q \,\big|\, Q \in \mathcal{M}_1(\mathcal{A})\}$$

- $\mathcal{M}_1(\mathcal{A})$ set of all probability measures on $\mathcal{A}$
- radius $r_n := r/\sqrt{n}$ shrinks with sample size $n \in \mathbb{N}$ where $r \in [0, \infty)$

Unique asymptotic MSE Solution

Theorem 5.5.7 (b), Rieder (1994)

$$\tilde{\eta}_\theta = (A_\theta \Lambda_\theta - a_\theta) w \qquad w = \min\left\{1, \frac{b_\theta}{|A_\theta \Lambda_\theta - a_\theta|}\right\}$$

with Lagrange multipliers A_θ, a_θ and b_θ determined by

$$0 = E_\theta(\Lambda_\theta - z_\theta) w \qquad a_\theta = A_\theta z_\theta \tag{1}$$

$$\mathbb{I}_k = A_\theta \, E_\theta(\Lambda_\theta - z_\theta)(\Lambda_\theta - z_\theta)^\tau w \tag{2}$$

$$r^2 b_\theta = E_\theta \left(|A_\theta \Lambda_\theta - a_\theta| - b_\theta\right)_+ \tag{3}$$

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Extension of classical Cramér-Rao bound

Proposition 2.1.1, Kohl (2005)

$$\text{(as.)maxMSE}(\eta_\theta, r) \geq \text{(as.)maxMSE}(\tilde{\eta}_\theta, r) = \operatorname{tr} A_\theta$$

Classical Cramér-Rao bound:

$$\mathrm{Cov}(\eta_\theta) \succeq \mathrm{Cov}(\hat{\psi}_\theta) = \mathcal{I}_\theta^{-1} \qquad \text{where } \hat{\psi}_\theta = \mathcal{I}_\theta^{-1}\Lambda_\theta$$

Hence

$$\mathrm{MSE}\,(\eta_\theta) = \operatorname{tr}\mathrm{Cov}(\eta_\theta) \geq \operatorname{tr}\mathrm{Cov}(\hat{\psi}_\theta) = \operatorname{tr}\mathcal{I}_\theta^{-1} = \mathrm{MSE}\,(\hat{\psi}_\theta)$$

Estimator Construction

One-step construction of optimally robust estimator $\tilde{S}_n$

$$\tilde{S}_n = \hat{\theta} + \frac{1}{n} \sum_{i=1}^{n} \tilde{\eta}_{\hat{\theta}}(y_i)$$

where $\hat{\theta}$ is a uniformly consistent starting estimator.

Examples for initial estimators: Kolmogorov or Cramér von Mises minimum distance estimators; cf. Rieder (1994).

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1. Asymptotic Theory of Robustness
   - Asymptotically Linear Estimators
   - Infinitesimal Neighborhoods
   - Optimally Robust Estimators
2. Convergence of Robust Models
   - Setup and Questions
   - Convergence of Robust Models

Setup

- Let

$$\mathcal{P}_\nu = \{P_{\nu,\theta} \,|\, \theta \in \Theta_\nu\} \subset \mathcal{M}_1(\mathcal{A}_\nu) \qquad \Theta_\nu \subset \mathbb{R}^k \text{ (open)}$$

  $(\nu \in \mathbb{N}_0)$ be a sequence of L_2-differentiable parametric families with L_2 derivatives $\Lambda_{\nu,\theta}$ and Fisher information of full rank $\mathcal{I}_{\nu,\theta}$.

- and consider

$$U_{\nu,c}(\theta, r_n) = \{(1 - r_n)_+ P_{\nu,\theta} + (1 \wedge r_n)\, Q_\nu \,\big|\, Q_\nu \in \mathcal{M}_1(\mathcal{A}_\nu)\}$$

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Questions

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Q1: $\mathcal{P}_\nu \approx \mathcal{P}_0$ for ν sufficiently large?

or even

Q2: $U_{\nu,c}(\theta_\nu, r_n) \approx U_{0,c}(\theta_0, r_n)$ for ν sufficiently large?

Q1: Convergence of Experiments; cf. Le Cam (1986)

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Q2: Assumptions

Assume

$$\mathcal{L}_{P_{\nu,\theta_\nu}}(\gamma_\nu^{-1} G_\nu \Lambda_{\nu,\theta_\nu}) \xrightarrow{\mathrm{w}} \mathcal{L}_{P_{0,\theta_0}}(\Lambda_{0,\theta_0}) \qquad \text{as } \nu \to \infty \tag{4}$$

$$\gamma_\nu^{-2}\, tr\, \mathcal{I}_{\nu,\theta_\nu} \longrightarrow tr\, \mathcal{I}_{0,\theta_0} \qquad \text{as } \nu \to \infty \tag{5}$$

for $(\gamma_\nu)_{\nu\in\mathbb{N}} \subset (0,\infty)$ and orthogonal $(G_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{R}^{k\times k}$.

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Q2: Convergence of Robust Models

Theorem 2.4.1, Kohl (2005)

Assume (4) *and* (5) *and denote the Lagrange multipliers contained in the MSE optimal solution* $\tilde{\eta}_{\nu,\theta_\nu}$ *by* A_{ν,θ_ν}, a_{ν,θ_ν} *and* b_{ν,θ_ν}. *Then,*

$$\lim_{\nu\to\infty} \gamma_\nu^2 \, tr\, A_{\nu,\theta_\nu} = tr\, A_{0,\theta_0} \qquad \lim_{\nu\to\infty} \gamma_\nu b_{\nu,\theta_\nu} = b_{0,\theta_0}$$

In case A_{0,θ_0} *and* a_{0,θ_0} *are unique, then also*

$$\lim_{\nu\to\infty} \gamma_\nu^2 G_\nu^\tau A_{\nu,\theta_\nu} G_\nu = A_{0,\theta_0} \quad \text{and} \quad \lim_{\nu\to\infty} \gamma_\nu G_\nu^\tau a_{\nu,\theta_\nu} = a_{0,\theta_0}$$

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Approximations

For ν sufficiently large, we have

$$
\begin{aligned}
\gamma_\nu^2 \, tr \, A_{\nu,\theta_\nu} \approx tr \, A_{0,\theta_0} \quad &\Longleftrightarrow \quad tr \, A_{\nu,\theta_\nu} \approx \gamma_\nu^{-2} \, tr \, A_{0,\theta_0} \\
\gamma_\nu b_{\nu,\theta_\nu} \approx b_{0,\theta_0} \quad &\Longleftrightarrow \quad b_{\nu,\theta_\nu} \approx \gamma_\nu^{-1} b_{0,\theta_0} \\
\gamma_\nu^2 G_\nu^\tau A_{\nu,\theta_\nu} G_\nu \approx A_{0,\theta_0} \quad &\Longleftrightarrow \quad A_{\nu,\theta_\nu} \approx \gamma_\nu^{-2} G_\nu A_{0,\theta_0} G_\nu^\tau \\
\gamma_\nu G_\nu^\tau a_{\nu,\theta_\nu} \approx a_{0,\theta_0} \quad &\Longleftrightarrow \quad a_{\nu,\theta_\nu} \approx \gamma_\nu^{-1} G_\nu a_{0,\theta_0}
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Note: To obtain an exact and not only an approximate IC we re-center and re-standardize the approximation.

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\gamma_\nu^2 G_\nu^\tau A_{\nu,\theta_\nu} G_\nu \approx A_{0,\theta_0} \quad &\Longleftrightarrow \quad A_{\nu,\theta_\nu} \approx \gamma_\nu^{-2} G_\nu A_{0,\theta_0} G_\nu^\tau\\
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\end{aligned}$$

Note: To obtain an exact and not only an approximate IC we re-center and re-standardize the approximation.

MSE-Inefficiency

Let χ be the re-centered and re-standardized approximating IC.

The MSE-Inefficiency of χ at radius $r \in [0, \infty)$ is defined as

$$relMSE_{\theta_0}(\chi, r) = \frac{E_{\theta_0} |\chi|^2 + r^2 [\sup_{P_{\theta_0}} |\chi|]^2}{tr\ A_{0,\theta_0}}$$

respectively as

$$relMSE_{\theta_\nu}(\chi, r) = \frac{E_{\theta_\nu} |\chi|^2 + r^2 [\sup_{P_{\theta_\nu}} |\chi|]^2}{tr\ A_{\nu,\theta_\nu}}$$

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Example 1: Normal Approximation of Binomial

Normal approximation: (“$mp(1-p) \geq 9$”)

$$\mathit{Binom}\,(m, p) \approx \mathcal{N}\left(mp, mp(1-p)\right)$$

We have $\nu = m$ and

$$\gamma_m = \sqrt{\frac{m}{p(1-p)}} \qquad \text{and} \qquad G_m = 1$$

cf. Lemma 3.2.2 in Kohl (2005).

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Poisson approximation: Let $\lambda = mp$ ("p small, m large"). Then,

$$Binom\,(m,p) \approx Pois\,(\lambda)$$

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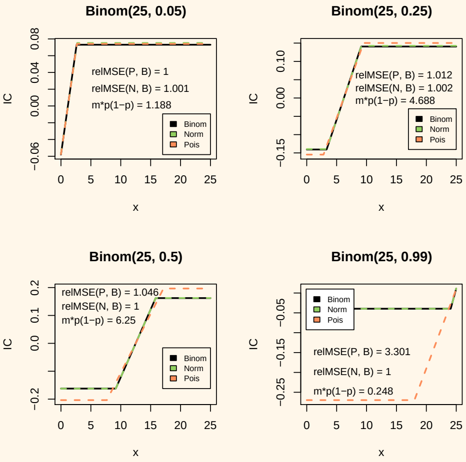
Examples 1+2: Approximation of ICs for $r = 0.25$



Example 3: Exponential Scale and Gumbel Location

It holds

$$\mathcal{L}_{Gum(0,1)}\big(-\Lambda^{loc}_{(0,1)}\big) = \mathcal{L}_{Exp(1)}\big(\Lambda^{sc}_{1}\big), \qquad \mathcal{I}^{loc}_{Gum(0,1)} = \mathcal{I}^{sc}_{Exp(1)}$$

i.e., $\gamma_\nu \equiv 1$ and $G_\nu \equiv -1$; cf. Section 5.2.1 in Kohl (2005).

Hence,

$$relMSE\,(\chi, r) \equiv 1$$

Note: Further examples are given in Kohl (2005).

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Summary

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- $\mathcal{P}_\nu \approx \mathcal{P}_0$
- arbitrary loss functions
- arbitrary estimators
- computation of distance?

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- $U_{\nu,c}(\theta_\nu, r_n) \approx U_{0,c}(\theta_0, r_n)$
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Outlook

- Extension to convex risks; cf. Ruckdeschel and Rieder (2004)
- Find/Consider further (multivariate) examples
- Application: If optimal IC is hard to compute in one model try to find an approximating model where the optimal IC is easier to compute and use it as approximation.

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Bibliography

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